ERROR IN THE NUMERICAL SOLUTION TO THE HEAT CONDUCTION PROBLEM WITH A NONLINEAR BOUNDARY CONDITION

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Concerning the heat conduction problem with a nonlinear boundary condition, as in the case of a boiling process, the effect of the time interval in the numerical solution scheme on the error of the solution is analyzed here.

The accuracy of the solution to heat conduction problems by the grid method depends on the length of time intervals ($\delta\tau$) and space intervals (h). The error incurred in finite-difference schemes has been estimated only qualitatively in theoretical analyses of this method of solution: its order of magnitude but not its actual value has been determined. In the practical sense, the accuracy depends not only on the size of $\delta\tau$ and h but also on the characteristics of the temperature field and on the temperature drops in time as well as in space. The characteristics of the field depend on the constraints and on how the thermophysical properties vary as functions of the temperature. It will be shown here that a drastic change in the boundary conditions, as during boiling, leads to large errors in a numerical solution with improperly selected time intervals.

The mathematical model of the phenomenon under study is

$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)-c_v\frac{\partial T}{\partial \tau}=0 \quad (0<x<R,\ \tau>0), \tag{1}$$

$$T(x,\ 0)=T_i, \tag{2}$$

$$\alpha(T_i-T_a)+\lambda\frac{\partial T(0,\ \tau)}{\partial x}=0, \tag{3a}$$

$$-\lambda\frac{\partial T(0,\ \tau)}{\partial x}=q(T_i)=f(T_i), \tag{3b}$$

$$\frac{\partial T(R,\ \tau)}{\partial x}=0. \tag{4}$$

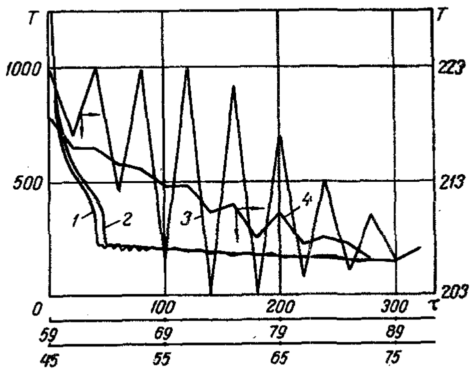

Fig. 1. Variation of surface temperature (T, °C) with time (τ, sec), based on various time intervals $\delta\tau$: variant 1 (1, 3, see Table 1), variant 2 (2, 4). T = 0-1400°C and τ = 0-300 sec (1, 2), T = 203-223°C and τ = 45-80 sec (3), T = 203-223°C and τ = 59-94 sec (4).

The problem is a nonlinear one with nonlinearities of the first kind $\lambda = \lambda(T)$, $c_v = c_v(T)$ as well as of the second kind $\alpha(T_s)$ or $q(T_s)$. Data on $\lambda(T)$ and $c_v(T)$ for the plate material (grade 08 steel) are given in the handbook [1]. We note that, within the test range of temperatures, both $\lambda(T)$ and $c_v(T)$ pass through an extremum, i.e., these temperature characteristics are by no means linear. The problem of cooling a plate with water was solved for a plate 2R = 0.16 m thick and at an initial temperature T_i = 1400°C, with the water at a constant temperature T_a = 20°C. The heat-transfer coefficient α or the thermal flux q at the surface during heat transfer through boiling attain their characteristic maximum during the critical boiling mode.

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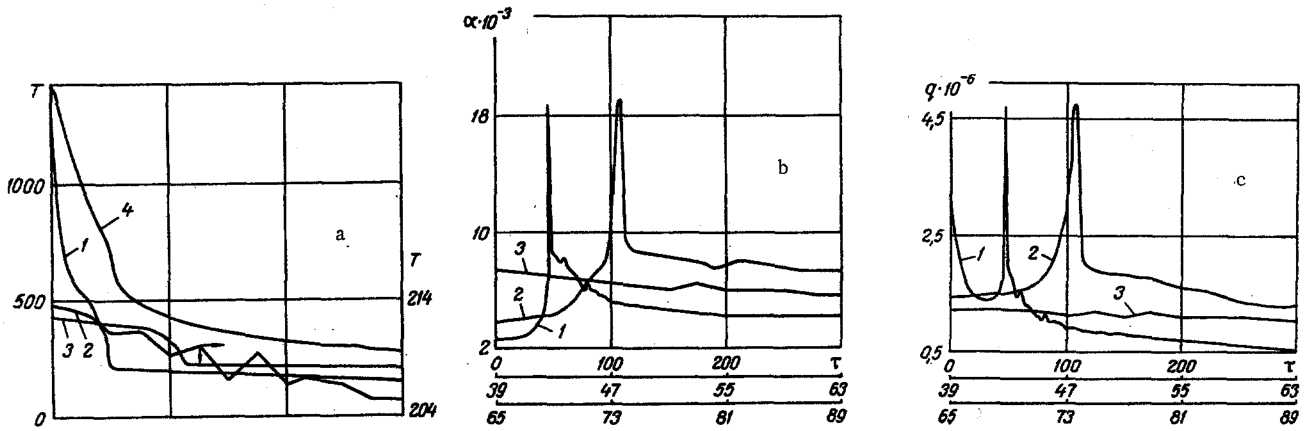

Fig. 2. a) Temperature at the surface x = 0 (1, 2, 3) and at point x = 0.01 m (4) as functions of time, according to variant 1 and with the temperature scale T = 0-1400°C for (1, 3, 4) and T = 204-214°C for 2. b) Heat-transfer coefficient α (kcal/m^2·h·deg)·10^{-3} as a function of time, according to variant 1. c) Thermal flux q (kcal/m^2·h)·10^{-6} as a function of time, according to variant 2. Times scales: τ = 0-300 sec (1, 4), 39-63 sec (2), 65-89 sec (3).

TABLE 1. Time Intervals $\delta\tau$ Chosen for Various Periods in the Solution Process

Time period sec	0—2	2—5	5—9	9—11	11—39	39—41	41—43	43—45	45—46	46—47	47—49
Variant 1	1	1	1	2	2	0,24	0,5	1	1	1	2
Variant 2	0,25	0,5	1	1	2	2	2	2	1	0,5	0,25
Time period sec	49—51	51—53	53—55	55—57	57—59	59—87	87—95	95——115	115——155	155——215	[illegible]——300
Variant 1	2	2	2	2	2	2	4	4	10	20	10
Variant 2	0,5	1	0,25	0,5	1	2	2	5	5	10	10

In order to obtain numerical results, we used an electrical model: a resistor grid suitable for a solution by the implicit finite-difference scheme [2]. The discreteness of the solution with respect to time and space made it easy to change the model parameters on each step, so as to account for the nonlinearities, by varying the time interval during the solution process.

In order to take into account the α(T) and the q(T) characteristics when this method is used [2], the electrical resistances, which either simulate the thermal resistances R_α to heat transfer at the surface or conduct current I_q simulating the thermal flux at the surface, are changed on each step of the solution in accordance with the respective surface temperature. When analyzing the effect of the boundary conditions, therefore, one may succeed equally well in simulating either the α(T) or the q(T) characteristic, it is also feasible to transfer from one to the other on the same electrical model.

The choice of the method was governed by the aim of the study, namely to establish the effect of drastic changes in α(T) or q(T) on the error in the solution using different time intervals. The space interval h was held constant and equal to R/8. Such an interval, without making the solution process more laborious, could yield an accuracy within the accuracy limit of the measuring circuit.* The time interval for the problem with such boundary conditions should be adjustable and its proper choice depends on α(T), q(T). The consequence of an error in determining T_s would be an incorrect value of α given for the respective step. With an improper (too long) time interval, the α(T) characteristic (its maximum) may have become blurred or altogether lost.

Variations in the surface temperature of the plate are shown in Fig. 1 corresponding to variants of the solution with improper and fixed time intervals as listed in Table 1.

*With an appropriate choice of $\delta\tau$ intervals, the error did not exceed 0.1%.

It is evident from the diagram that the consequence of longer time intervals (see variant 1) are amplitudes of temperature fluctuations at the plate surface. In variant 2 (curves 2 and 4) these amplitudes are 5-6 times smaller than in variant 1 (curves 1 and 3). Choosing longer time intervals during rapid changes in surface temperature (see curves 1 and 2 during the 35-55 sec period) leads to large errors in determining the temperatures as well as the instants of time at which those temperature changes occur. At $\tau = 45$ sec (see Fig. 1), for example, $T_S = 223°C$ according to variant 1 and $T_S = 385°C$ according to variant 2; the respective errors are $\Delta = -11.5\%$ and $\delta = 42\%$. The time to reach a temperature of, say, 350°C differs by 17% between the two variants of the solution. This example illustrates the importance of choosing the proper time interval o in the solution of nonlinear problems with drastic changes in $\alpha(T, \tau)$ or $q(T, \tau)$. All arbitrary choice of $\delta\tau$ leads to large errors in the determination of absolute temperature, temperature drops, thermal fluxes, and the run-in time after which a heat apparatus becomes operative.

It is to be noted that, in both variants of the solution, the widest temperature fluctuations appear at the surface nodes of the simulating grid, while already at the nearest to the surface nodes ($x = 0.01$ m away) fluctuations have ceased almost entirely. The curves in Fig. 2 represent the time-variation of the surface temperature, the temperature at point $x = 0.01$ m, the heat-transfer coefficient α, and the thermal flux q. The values of α and q have been plotted as functions of time, in order to show how they change in the course of the solution process.

In order to plot $\alpha(\tau)$ and $q(\tau)$, it is necessary to solve $\alpha[T_S(\tau)]$ and $q[T_S(\tau)]$. If more accurate data are to be obtained when α and q change rapidly, then the time intervals must be reduced. The fluctuations of α and q (see curves 2, 3 in Fig. 2b, c) indicate that choosing shorter time intervals $\delta\tau$ in variant 2 has not eliminated fluctuations of T_S. The fluctuations of T_S in variant 2 are smaller than in variant 1, but some slight fluctuations of α and q still occur. The evidence must weighed in the solution of reverse problems (linear problems with variable $\alpha(\tau)$, $q(\tau)$, or nonlinear problems).

The solution of reverse and inverse problems (the solution to which is, apparently, stable in the small and unstable in the large) requires a more careful choice of h and α than the solution of forward problems. In forward problems the choice of h and α determines only the accuracy of the results, the qualitative conclusions remain valid anyway (when implicit schemes of solution are used). In reverse and inverse problems an improper choice of h and $\delta\tau$ may lead not only to large quantitative errors but also to a qualitative misrepresentation of the sought characteristics.

NOTATION

T	is the temperature;
τ	is the time;
λ	is the thermal conductivity;
c_V	is the volumetric specific heat;
α	is the heat-transfer coefficient;
q	is the thermal flux density;
h	is the space interval;
$\delta\tau$	is the time interval;
$\Delta = (T_{i1} - T_{i2})/(T_{max})\,100\%$	is the error with respect to maximum temperature;
$\delta = (T_{i1} - T_{i2})/(T_{i2})\,100\%$	is the error with respect to instantaneous temperature.

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